

Pseudogap and time reversal breaking in a holographic superconductor

Matthew M. Roberts

*Department of Physics, University of California,
Santa Barbara, CA 93106-9530, U.S.A.
E-mail: matt@physics.ucsb.edu*

Sean A. Hartnoll

*KITP, University of California,
Santa Barbara, CA 93106-4030, U.S.A.
E-mail: hartnoll@kitp.ucsb.edu*

ABSTRACT: Classical $SU(2)$ Yang-Mills theory in 3+1 dimensional anti-de Sitter space is known to provide a holographic dual to a 2+1 system that undergoes a superconducting phase transition. We study the electrical conductivity and spectral density of an isotropic superconducting phase. We show that the theory exhibits a pseudogap at low temperatures and a nonzero Hall conductivity. The Hall conductivity is possible because of spontaneous breaking of time reversal symmetry.

KEYWORDS: AdS-CFT Correspondence, Spontaneous Symmetry Breaking, Discrete and Finite Symmetries.

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1. Introduction

The first successful microscopic theory of superconductivity, BCS theory [1], was developed over fifty years ago and correctly describes the superconducting phenomenology of a large number of metals and alloys [2]. The essence of superconductivity is the spontaneous breaking at low temperatures of a $U(1)$ symmetry due to a charged condensate. In BCS theory, the condensate is a Cooper pair of electrons, bound together by lattice vibrations, or phonons.

It has been appreciated for some time that materials of significant theoretical and practical interest, such as the heavy fermion compounds [3, 4] or the high T_c cuprates [5], require new theoretical input. For these materials, neither the pairing mechanism, leading to the charged condensate, nor the properties of the superconducting state itself are those of BCS theory. Furthermore, there are indications that the relevant new physics is strongly coupled, requiring a departure from the quasiparticle paradigm of Fermi liquid theory [3, 5].

Our hope is that a solvable model of a strongly coupled system undergoing a superconducting phase transition might help the development of new theories of superconductivity. It has recently been shown that the AdS/CFT correspondence [6] can indeed provide models of strongly interacting superconductors in which calculations can be performed from first principles [7–10]. These recent works are part of a wider program of applying the AdS/CFT correspondence to condensed matter systems [11–16]. The philosophy is that even if the underlying microscopic descriptions of theories with AdS duals are likely quite different to those arising in materials of experimental interest, aspects of the strongly coupled dynamics and kinematics may be universal. Kinematically speaking, theories with AdS duals are quantum critical [17]. The superconductors described to date within the AdS/CFT framework are quantum critical systems that undergo a superconducting phase transition as a function of temperature over chemical potential.

In the recent work [9], the electrical conductivity of a holographic (i.e. AdS/CFT) superconductor was computed as a function of frequency. A delta function at the origin, $\omega = 0$, due to the Goldstone boson of the broken $U(1)$ symmetry, was followed by a gap

2Δ in which the dissipative conductivity vanished at zero temperature. At small but finite temperature the conductivity in the gap was suppressed by $e^{-\Delta/T}$. Beyond the gap, a finite spectral density was observed, exhibiting or not a coherence peak depending on the details of the system. What was remarkable about these results was that despite coming from a strongly coupled quantum critical theory, they are exactly the qualitative features of conductivity in the superconducting phase that one obtains from weakly coupled BCS theory [2]. The main difference, and apparently only indication of an underlying strong coupling dynamics, was that [9] found the zero temperature gap $2\Delta/T_c \approx 8.4$ rather than $2\Delta/T_c \approx 3.5$ for BCS theory.

It is of interest to find and study gravitational duals to superconductivity with qualitative features that are not those of BCS theory, but rather of nonconventional superconductors such as the heavy fermions or the cuprates. In this paper we will study the electrical conductivity of a holographic superconductor recently proposed by Gubser [8]. We show that this model exhibits two such nonconventional features: a ‘pseudogap’ rather than a gap at zero temperature and spontaneous breaking of time reversal invariance.

By ‘pseudogap’ we do not mean the exotic and controversial region in the normal phase of cuprate superconductors [5]. Rather, we will use the term to denote a well-defined gap in the dissipative conductivity at low frequencies in which the conductivity is not identically zero. In nonconventional superconductors, this pseudogap is due to the fact that the Cooper pairs are not bound states with zero angular momentum ($l = 0$) but rather so-called p -wave ($l = 1$) or d -wave ($l = 2$ spin singlet) states. The gap above the Fermi surface in these superconductors vanishes at certain specific directions in momentum space and therefore one finds (a reduced number of) excitations with arbitrarily low energy [5, 4].

Spontaneous breaking of time reversal invariance has recently been observed, for instance, in the YBCO high T_c superconductor [18–22]. The breaking is thought to occur because the condensing Cooper pairs are not only not s -waves but in fact a complex combination of d -waves: $d_{x^2-y^2} + id_{xy}$. The simple fact that this is a superposition of T invariant states with differing phases is sufficient to break T invariance. Recall that T is an anti-linear operator. One immediate consequence of breaking time reversal is that it is possible for these systems to have a Hall conductivity even in the absence of an external magnetic field [23, 24]. We will review this fact below.

The layout of this paper is as follows. We will first review the Einstein-Yang-Mills system that was shown in [10] to be a holographic dual to a theory with a superconducting phase transition that spontaneously breaks T invariance. We will compute the electric conductivity and the spectral density in the superconducting phase and exhibit a pseudogap and a Hall conductivity.

Note: as this paper was nearing completion, the preprint [25] was posted to the arxiv, with some overlap with this work. Furthermore, it was pointed out in [25] that the superconducting phase we will be studying is in fact unstable near T_c to the breaking of rotational invariance. We have extended and confirmed this stability analysis in section 4 below. The instability is in a mode orthogonal to the modes we study. Therefore our results for the conductivity should be interesting as a strong coupling computation of an

isotropic pseudogapped superconducting phase with both Hall and direct conductivities (with no external magnetic field).

2. The Einstein-Yang-Mills background

The gravitational geometry dual to our finite temperature superconductor will be the planar Schwarzschild-AdS black hole in 3+1 dimensions

$$ds^2 = \frac{r^2}{L^2} \left(-h(r)dt^2 + 2dzd\bar{z} \right) + \frac{L^2}{r^2 h(r)} dr^2, \quad (2.1)$$

where $h(r) = 1 - \frac{r_0^3}{r^3}$ and we have introduced the complex coordinates

$$z = \frac{x + iy}{\sqrt{2}}. \quad (2.2)$$

The scale L is the AdS radius and r_0 gives the location of the black hole horizon. The geometry (2.1) is a solution of the vacuum Einstein equations with a negative cosmological constant.

On this background we will study an SU(2) gauge theory. This was suggested to be an interesting dual for a superconductor in [10], and we shall review the dynamics shortly. The full Einstein-Yang-Mills theory has the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa_4^2} \left(R + \frac{6}{L^2} \right) - \frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \right]. \quad (2.3)$$

Following [9, 10], we will be working in the probe limit, in which the SU(2) fields are small and do not backreact on the metric.¹ Conformal invariance of the Yang-Mills action means that by rescaling the metric, keeping the Yang-Mills coupling g fixed, we can always make the probe approximation consistently. Specifically, we take $Lg \gg \kappa_4$. In fact, by rescaling all the fields we can set $L = 1$, which we will proceed to do. Let us now set up some notation.

The three generators τ^a of the SU(2) algebra satisfy $[\tau^b, \tau^c] = \tau^a f^{abc}$. In the standard basis we have $f^{abc} = \epsilon^{abc}$, with $\epsilon^{123} = 1$. It will be useful for us to work with the combinations

$$\tau^\pm = \frac{\tau^1 \pm i\tau^2}{\sqrt{2}}, \quad (2.4)$$

as well as τ^3 . The field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \quad (2.5)$$

The equations of motion are

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{a\mu\nu}) + f^{abc} A_\mu^b F^{c\mu\nu} = 0. \quad (2.6)$$

¹The probe limit is in fact necessary to obtain a finite DC ($\omega = 0$) conductivity even in the normal phase. A translationally invariant system with a finite charge density will have an infinite DC conductivity unless the momentum can leak somewhere. In the probe limit, momentum is leaked to the metric or ‘glue’ sector. See for instance [26].

Following [10] we will identify the U(1) subgroup of SU(2) generated by τ^3 to be the electromagnetic U(1). The W-bosons A^\pm are therefore charged fields. The insight of [10] was to note that a sufficiently large background electric field for the U(1), at fixed temperature, would cause the W-bosons to condense and hence trigger superconductivity. Specifically, consider the background ansatz

$$A = \phi(r) dt\tau^3 + w(r) (dz\tau^- + d\bar{z}\tau^+). \quad (2.7)$$

An important feature of this ansatz is that whereas the original rotational invariance is broken, the system is invariant under a combined gauge and spatial rotation

$$z \rightarrow e^{i\theta} z, \quad \tau^\pm \rightarrow e^{\pm i\theta} \tau^\pm. \quad (2.8)$$

Therefore the superconducting phase we are considering is effectively spatially isotropic. A symmetry that is broken by the $A_{z,\bar{z}}^\pm(r)$ fields is time reversal invariance. These gauge potentials lead to a magnetic field in the bulk. A magnetic field breaks time reversal, as can be immediately seen from (for instance) the Lorentz force law.

Evaluated on this ansatz, the equations of motion become

$$\begin{aligned} \phi'' + \frac{2}{r}\phi' - \frac{2w^2}{r^4 h}\phi &= 0, \\ w'' + \left(\frac{2}{r} + \frac{h'}{h}\right)w' + \frac{\phi^2 w - hw^3}{r^4 h^2} &= 0. \end{aligned} \quad (2.9)$$

Under a rescaling $r = r_0 \bar{r}$, $\phi(r) = r_0 \bar{\phi}(\bar{r})$, $w(r) = r_0 \bar{w}(\bar{r})$, the black hole radius r_0 cancels out of the equations entirely, rendering all quantities dimensionless. This is a consequence of the dual theory being a conformal field theory in flat space. Therefore we will set $r_0 = 1$ in what follows. The Hawking temperature of the black hole (2.1) is

$$T = \frac{3r_0}{4\pi}. \quad (2.10)$$

We will restore the r_0 dependence when we wish to re-express quantities in terms of the temperature.

There are two solutions to the equations (2.9). The first is the AdS-Reissner-Nordstrom black hole which has²

$$\phi = \rho(1 - 1/r), \quad w = 0. \quad (2.11)$$

The scalar potential is required to vanish on the horizon (recall that we have set $r_0 = 1$). This ties the chemical potential μ to equal the charge density ρ . The second solution is a hairy black hole. This solution needs to be found numerically. The main feature is that $w \neq 0$, but still normalizable at infinity. Thus at large radius r we require for the scalar potential

$$\phi = \mu - \frac{\rho}{r} + \dots, \quad (2.12)$$

²In the following definitions we have rescaled the boundary charge density and condensate by $\rho, J \rightarrow \rho/g^2, J/g^2$, to eliminate messy factors of g . This rescaling does not affect the physical ratio $2\Delta/T_c$.

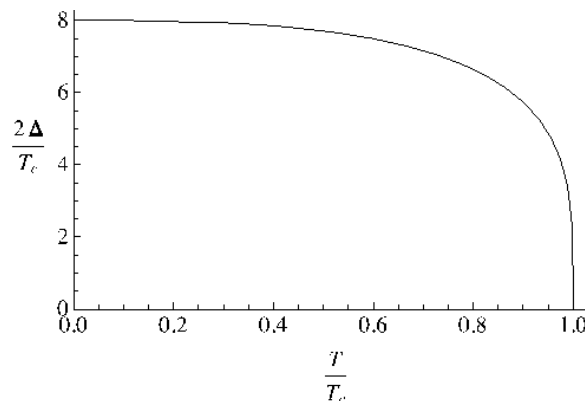


Figure 1: The gap as a function of temperature.

and for the charged condensate

$$w = \frac{\langle J \rangle}{\sqrt{2r}} + \dots \quad (2.13)$$

Here $\langle J \rangle$ is a condensate of the charged operator dual to $A_z^- \tau^- + A_z^+ \tau^+$ (up to a sign). It is a component of the global $SU(2)$ current in the field theory. We have required that there is no source term in field theory action for the operator J , by demanding that the constant term in the large r expansion of w vanish. From the field theory point of view, the presence of the $\langle J \rangle$ condensate in the absence of a source means that time reversal symmetry is spontaneously broken.

In order to find the hairy black hole solutions, one should numerically integrate out from the horizon. Near the horizon one writes $\phi = \phi_1(r-1) + \dots$ and $w = w_0 + \dots$. There are two constants of integration, w_0 and ϕ_1 , and therefore, upon imposing the falloff (2.13) we are left with a one parameter family of solutions.

If we fix the charge density ρ , one finds that the hairy black holes only exist below a critical temperature

$$T_c = 0.125\sqrt{\rho}. \quad (2.14)$$

See figure 1. With some hindsight, see the following section, we introduce gap notation

$$2\Delta = \sqrt{\langle J \rangle}. \quad (2.15)$$

Note that J is an operator with mass dimension 2, appropriate for a current density in $2+1$ dimensions. The natural dimensionless quantities to plot are therefore $2\Delta/T_c$ against T/T_c . Just below the critical temperature we find

$$\langle J \rangle \approx 104.8T_c^2(1 - T/T_c)^{1/2}, \quad (2.16)$$

as expected for a mean field second order transition. It is interesting to note that the zero temperature gap is

$$\frac{2\Delta(0)}{T_c} \approx 8. \quad (2.17)$$

This is greater than the BCS value, rather close to the value found in the holographic model of [9] and consistent with some values observed in the cuprates. Given the similarity of this value for the gap and the one obtained in [9], it would be of interest to obtain values for different holographic models to see to what extent it is universal.

Before moving on we should make some comments concerning the zero temperature limit. We see in figure 1 and in the figures below that all the quantities we consider converge nicely when we take $T \ll T_c$. However, it is not straightforward to extract the limiting results by studying (zero temperature) AdS directly. This point was noted and discussed in [9]. Understanding how to treat holographic superconductors directly at zero temperature is an important question for future work. Also as noted in [9], the strict zero temperature limit does not commute with the probe approximation. The probe limit may, however, be self consistently used to arbitrarily low temperatures. Work in going beyond the probe approximation is underway [27].

3. Conductivity and spectral density

Two basic quantities characterising any superconductor are the frequency dependent conductivity and the spectral density. These are closely related, but not identical in the presence of a Hall conductivity. The electrical conductivity is defined through Ohm's law

$$J_i = \sigma_{ij} E^j. \tag{3.1}$$

Here E^j is an external electric field and J_i the current generated. Because the superconducting phase we are studying is isotropic, we will have $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$. In a non-isotropic phase the distinction between the standard and Hall conductivities is not well defined.

We will compute the conductivity directly from Ohm's law (3.1). However, to relate the conductivity to the spectral density, we note the standard formula from linear response theory

$$\sigma_{ij}(\omega) = \frac{-iG_{ij}^R(\omega)}{\omega}, \tag{3.2}$$

in which the retarded Greens function is

$$G_{ij}^R(\omega) = -i \int d^2x dt e^{-i\omega t} \theta(t) \langle [J_i(t), J_j(0)] \rangle. \tag{3.3}$$

The spectral density is defined to be twice the imaginary part of the retarded Greens function. However, in order to obtain a sensible, i.e. non-negative, spectral density one needs the Greens function of currents that do not couple. These are obtained by diagonalising the conductivity. The conductivity matrix has eigenvalues

$$\sigma_z = \sigma_{xx} + i\sigma_{xy}, \quad \sigma_{\bar{z}} = \sigma_{xx} - i\sigma_{xy}. \tag{3.4}$$

In the eigenbasis Ohm's law becomes the decoupled equations $J_z = \sigma_z E_z$ and $J_{\bar{z}} = \sigma_{\bar{z}} E_{\bar{z}}$. We are using the complex coordinates introduced in (2.2) above. This complexified formalism is similar to that used in [13]. The spectral densities associated with the decoupled

currents J_z and $J_{\bar{z}}$ are thus

$$\chi_z(\omega) = 2\omega \text{Re } \sigma_z(\omega), \quad \chi_{\bar{z}}(\omega) = 2\omega \text{Re } \sigma_{\bar{z}}(\omega). \quad (3.5)$$

We recall that the physical interpretation of the spectral density is that it gives us the density of energy eigenstates at energy ω , weighted by their overlap with the electric current operators. Therefore the spectral density is the appropriate quantity with which to probe the pseudogap region, that will make an appearance shortly.

In order to compute linear response functions, such as the conductivity, using the gravitational dual to the superconductor, one needs to consider linearised perturbations of the fields about the black hole background [28]. Specifically, we are interested in the τ^3 component of the nonabelian current, $j_{x,y}^3$. This will be dual to fluctuations of the $A_{x,y}^3$ fields. Because of the nonlinearities in the Yang-Mills action, fluctuations in $A_{x,y}^3$ will source other fields. We need to keep all the modes that are coupled at a linearised level for consistency.

Since our background is invariant under a combined spatial and gauge rotation (2.8), it is sufficient to consider fields that have the same charge as $A_{x,y}^3$ under this $U(1)$ action. Specifically, we see that there will be decoupled equations involving the sets of fields

$$\{A_r^+(r), A_t^+(r), A_z^3(r)\} \quad \text{and} \quad \{A_r^-(r), A_t^-(r), A_z^3(r)\}. \quad (3.6)$$

All of these fields are taken to have an overall time dependence of $e^{-i\omega t}$.

There is still some gauge freedom left, which reduces the actual degrees of freedom. In particular, we can consider a background field gauge transformation generated by some λ^a

$$\delta_{BG} A_\mu^a = \partial_\mu \lambda^a + f^{abc} A_\mu^b \lambda^c. \quad (3.7)$$

We will use this freedom to set $A_r^\pm = 0$. The linearized equations for the perturbations become

$$A_z^{3''} + \left(\frac{2}{r} + \frac{h'}{h}\right) A_z^{3'} + \frac{\omega^2 - hw^2}{r^4 h^2} A_z^3 - \frac{w(\phi + \omega)}{r^4 h^2} A_t^+ = 0, \quad (3.8)$$

$$A_{\bar{z}}^{3''} + \left(\frac{2}{r} + \frac{h'}{h}\right) A_{\bar{z}}^{3'} + \frac{\omega^2 - hw^2}{r^4 h^2} A_{\bar{z}}^3 - \frac{w(\phi - \omega)}{r^4 h^2} A_t^- = 0, \quad (3.9)$$

$$A_t^{+''} + \frac{2}{r} A_t^{+'} - \frac{w^2}{r^4 h} A_t^+ + \frac{w(\phi + \omega)}{r^4 h} A_z^3 = 0, \quad (3.10)$$

$$A_t^{-''} + \frac{2}{r} A_t^{-'} - \frac{w^2}{r^4 h} A_t^- + \frac{w(\phi - \omega)}{r^4 h} A_{\bar{z}}^3 = 0, \quad (3.11)$$

$$A_t^+ \phi' + h(w' A_z^3 - w A_z^{3'}) - (\phi - \omega) A_t^{+'} = 0, \quad (3.12)$$

$$A_t^- \phi' + h(w' A_{\bar{z}}^3 - w A_{\bar{z}}^{3'}) - (\phi + \omega) A_t^{-'} = 0. \quad (3.13)$$

Two of these equations are redundant: Equations (3.10) and (3.11) can be derived by differentiating the first order equation and substituting the other equations of motion. Therefore, in solving these equations numerically it is sufficient to numerically integrate (3.8), (3.9), (3.12) and (3.13). Despite the first-derivative equations appearing to have singular points, they numerically integrate perfectly well.

The equations are solved numerically by integrating out from the horizon to infinity. In order to obtain retarded Greens functions, ingoing boundary conditions must be imposed at the horizon [28]. This is most conveniently done by requiring the following behaviour near the horizon $r \approx 1$:

$$A_z^3 = h^{-i\omega/3} a_0 + \dots, \tag{3.14}$$

$$A_{\bar{z}}^3 = h^{-i\omega/3} b_0 + \dots, \tag{3.15}$$

$$A_t^+ = \frac{a_0 w_0}{i + \omega/3} h^{-i\omega/3} (r - 1) + \dots, \tag{3.16}$$

$$A_t^- = -\frac{b_0 w_0}{i + \omega/3} h^{-i\omega/3} (r - 1) + \dots. \tag{3.17}$$

Recall that we defined w_0 above as the value of the background field w at the horizon. Given the background, there are two free constants, a_0 and b_0 .

Integrating the fields out to large r , we can read off the dual currents and external electric fields. The current and charge densities are obtained from

$$F_{r\mu}^a = \frac{g^2 \langle J_\mu^a \rangle}{r^2} + \dots, \tag{3.18}$$

where μ here runs over the boundary directions t, z, \bar{z} . In this expression we have included the Yang-Mills coupling g due to the action (2.3). This coupling determines the (constant) conductivity of the normal, non-superconducting, state [11]

$$\sigma_n = \frac{1}{g^2}. \tag{3.19}$$

The numerical value of the normal state conductivity depends on the theory. The external electric fields then are obtained from

$$F_{ti}^a = -E_i^a + \dots. \tag{3.20}$$

(A more symmetric formulation of (3.18) and (3.20) is possible in terms of the radial variable $u = 1/r$.) From the previous section, we have the background equilibrium values

$$\langle J_{z,\bar{z}}^\pm \rangle = -J, \quad \langle J_t^3 \rangle = \rho. \tag{3.21}$$

In reading off the linearised electric response to a time varying external field we have to face the fact that the naïve SU(2) conductivity, or the projection of it onto the τ^3 direction, is not SU(2) invariant. We are interested in the electrical conductivity of the U(1) subgroup of SU(2) generated by τ^3 . Therefore, we should consider currents that result from external sources in the τ^3 direction only. In many circumstances it would be sufficient to ensure that we have electric field $E_i \equiv E_i^3$ only. However, because the naïve rotational invariance is broken, there is a nontrivial charge-current density Green's function $G_{it}^R(\omega)$. Therefore, we need a configuration in which the background A_t^\pm also vanishes. Here A_t^\pm is a source for the charge density ρ^\pm .

If we were to simply take our equations (3.8) to (3.13) and integrate them to the boundary, while we would never obtain electric fields E_i^\pm , we would obtain a source A_t^\pm .

We therefore need to do an SU(2) rotation to set this term to zero. In the bulk this means we should allow for a gauge transformation that sets the boundary value of A_t^\pm to zero. A gauge transformation generated by $\lambda^a(r)$ results in the new scalar potential

$$\delta A_t^\pm = A_t^\pm - i\lambda^\pm(\omega \mp \phi), \quad (3.22)$$

as well as the new field strengths

$$\delta F_{rt}^\pm = A_t^{\pm'} \pm i\lambda^\pm \phi', \quad (3.23)$$

$$\delta F_{rz}^3 = A_z^{3'} - i\lambda^+ w', \quad (3.24)$$

$$\delta F_{r\bar{z}}^3 = A_{\bar{z}}^{3'} + i\lambda^- w', \quad (3.25)$$

$$\delta F_{tz}^3 = i\omega A_z^3 + w(iA_t^+ \mp \lambda^+ \phi). \quad (3.26)$$

In the final of these expressions, only the first term contributes at the boundary $r \rightarrow \infty$.

We wish to cancel the leading asymptotic term in A_t^\pm with λ . After doing this, we can obtain the conductivities

$$\sigma_z = \frac{J_z}{E_z} = -\sigma_n \lim_{r \rightarrow \infty} \frac{r^2 \delta F_{rz}^3}{\delta F_{tz}^3} = \sigma_n \lim_{r \rightarrow \infty} \frac{i}{\omega A_z^3} \left(r^2 A_z^{3'} + \frac{J}{\sqrt{2}(\omega - \mu)} A_t^+ \right), \quad (3.27)$$

$$\sigma_{\bar{z}} = \frac{J_{\bar{z}}}{E_{\bar{z}}} = -\sigma_n \lim_{r \rightarrow \infty} \frac{r^2 \delta F_{r\bar{z}}^3}{\delta F_{t\bar{z}}^3} = \sigma_n \lim_{r \rightarrow \infty} \frac{i}{\omega A_{\bar{z}}^3} \left(r^2 A_{\bar{z}}^{3'} - \frac{J}{\sqrt{2}(\omega + \mu)} A_t^- \right). \quad (3.28)$$

We can see that this expression has a curious pole at $\omega = \mu$. This will appear as a delta function in the spectral density for J_z .³ It is of interest to elucidate the physics behind this stable resonance, but we shall not do so here.

Given σ_z and $\sigma_{\bar{z}}$ we can obtain the spectral functions directly from (3.5) and the standard and Hall conductivities by inverting (3.4) to obtain

$$\sigma_{xx} = \frac{1}{2}(\sigma_z + \sigma_{\bar{z}}), \quad \sigma_{xy} = \frac{-i}{2}(\sigma_z - \sigma_{\bar{z}}). \quad (3.29)$$

Our result for the spectral functions is plotted in figure 2. The standard and Hall conductivities are plotted in figure 3.

The principal qualitative features of our results are immediately seen in these plots. Firstly, there is a strong depletion of spectral weight at small frequencies. However, even as we take the temperature to zero, the spectral function does not become identically zero over a finite range of low frequencies. Therefore, we will call this region the pseudogap. The two currents, J_z and $J_{\bar{z}}$ have pseudogaps of differing widths. We have defined 2Δ to be the width of the narrower of the two, see the right hand plot of figure 2. Obviously there is some ambiguity in the definition of the width. The two different gaps are likely due to the fact that there are effectively two condensates in the ansatz (2.7). It would be interesting to disentangle their physics by considering more general backgrounds with more than one radial function. Allowing for gauge transformations, (2.7) can be generalised to three independent radial functions. The only isotropic ansatz is (2.7), up to gauge transformations.

³We thank Gubser and Pufu for convincing us that this pole is indeed physical.

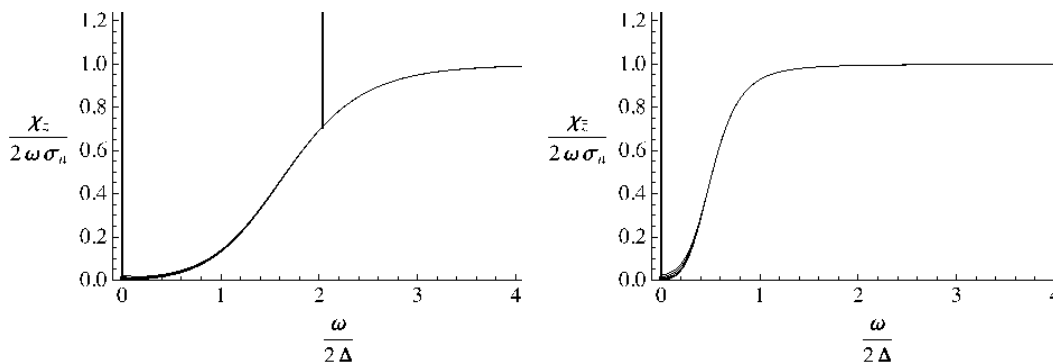


Figure 2: Spectral functions for the currents J_z and $J_{\bar{z}}$. Each plot is in fact five curves, at temperatures $T/T_c = .08, .11, .15, .19, .23$. We see that we have effectively reached the zero temperature limit. There is a clear pseudogap, and a delta function at $\omega = 0$. There is also a delta function at $\omega = \mu$.

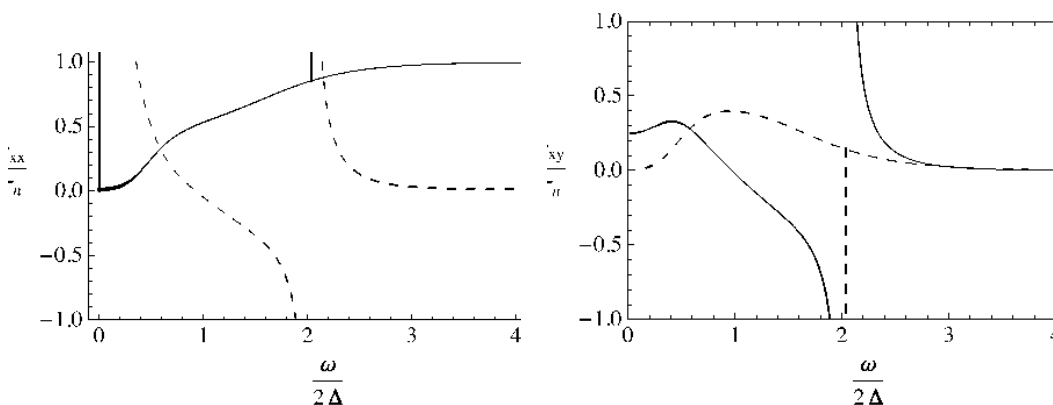


Figure 3: Standard and Hall conductivities at low temperatures as a function of frequency. The temperatures shown are again $T/T_c = .08, .11, .15, .19, .23$. The solid lines are the real part whereas the dashed lines are the imaginary parts. A pole in the imaginary part at $\omega = 0$ indicates that the real part will contain a delta function at $\omega = 0$. Similarly with the pole at $\omega = \mu$

In the standard conductivity we find a delta function in the real part at $\omega = 0$. This is the Goldstone boson of the spontaneously broken $U(1)$ symmetry and is the signal of superconductivity. We again see a pseudogap at low frequencies. Because this conductivity is sensitive to both the χ_z and $\chi_{\bar{z}}$ spectral functions, which had gaps of differing widths, there is a small feature in the conductivity.

We also find a nonvanishing Hall conductivity. This Hall conductivity is different to previous appearances of Hall conductivity in AdS/CFT, e.g. [29], in that it is not due to an external magnetic field. Rather, a Hall conductivity is possible because the superconducting condensate broke time reversal invariance. This can be seen directly from (3.2) and (3.3). Recalling that T is an antilinear operator, then (3.3) implies that $\sigma_{ij}(\omega) = \sigma_{ji}(\omega)$ in a

state that is T-invariant.⁴ Isotropy implies $\sigma_{xy}(\omega) = -\sigma_{yx}(\omega)$, and therefore T-invariance implies $\sigma_{xy}(\omega) = 0$. Furthermore (recall that the conductivity is dimensionless in 2+1 dimensions) we find that the Hall conductivity attains a finite real value at $\omega = 0$.

Now let us extract some quantitative results from our data. The superfluid dens is the coefficient of the delta function in the real conductivity at $\omega = 0$. By the Kramers-Kronig relations, this is also the coefficient of the pole in the imaginary part of the conductivity as $\omega \rightarrow 0$:

$$\text{Re } \sigma_{xx}(\omega) \sim \pi n_s \delta(\omega) \quad \rightarrow \quad \text{Im } \sigma_{xx}(\omega) \sim n_s / \omega. \quad (3.30)$$

From our equations we find

$$\text{for } T \ll T_c : \quad n_s \approx C \sigma_n \Delta, \quad (3.31)$$

$$\text{as } T \rightarrow T_c : \quad n_s \approx C' \sigma_n (T_c - T), \quad (3.32)$$

where the numerical coefficients are $C = 0.25$, $C' = 12.2$. As in [9], we can note that a linearly vanishing superfluid density near the critical temperature results in a London magnetic penetration depth of $\lambda_L \sim (T_c - T)^{-1/2}$, as expected from Landau-Ginzburg theory.

We can also read off the ‘normal component’ of the DC conductivity, that is, the finite DC conductivity due to finite temperature excitations of the normal state. In contrast to the fully gapped model of [9], there is no exponential suppression at small temperatures. Instead we find a quadratic dependence on temperature

$$n_n \equiv \lim_{\omega \rightarrow 0} \text{Re } \sigma_{xx} \approx 0.32 \sigma_n (T/T_c)^2 + \dots, \quad (3.33)$$

and in the zero temperature limit the small frequency dependence is

$$\text{Re } \sigma_{xx} \approx 0.20 \sigma_n (\omega/2\Delta)^2 + \dots. \quad (3.34)$$

Similarly, we can examine the DC hall effect and find

$$H \equiv \lim_{\omega \rightarrow 0} \text{Re } \sigma_{xy} \approx 0.24 \sigma_n + 0.13 \sigma_n (T/T_c)^2 + \dots. \quad (3.35)$$

There is no superfluid component because the imaginary part has no pole. On the other hand, because there is no Hall conductivity in the normal phase, it is clear that superconducting physics is playing an important role.

The crucial question is of course to understand the physics underlying the pseudogap in this model. One might hope that the psuedogap is indeed a signal of the p - or d -wave nature of the ‘Cooper pairs’. The operator that condenses is a component of a global SU(2) current. In many cases this will be a bilinear in UV operators, including fermions. The combination of spacetime and internal symmetries that occurs in these models to preserve isotropy is also reminiscent of non s -wave superconductors. In order to be completely sure that there is indeed a non s -wave condensate one should probe the system as a function

⁴ $\langle [J_i(t), J_j(0)] \rangle = \langle T [J_i(t), J_j(0)] T^* \rangle = \langle [J_j(0), J_i(-t)] \rangle = \langle [J_j(t), J_i(0)] \rangle$.

of momentum k_i and look for a momentum-dependent gap. We hope to address this in the future.

Another possibility is that the pseudogap is directly due to the fact that there is a massless excitation in the theory: the Goldstone boson.⁵ The electric current can decay at arbitrarily small energy into a multi Goldstone boson state. By the optical theorem, this process should introduce a branch cut in the current-current Greens function reaching down to $\omega = 0$. In fact, one might ask why such a cut didn't show up in the conductivity computations in [9]. At weak coupling such processes are at higher loop order and are suppressed relative to pair production of 'electrons' (the quasiparticles forming the Cooper pair). At strong coupling it is less clear. Perhaps such cuts reaching $\omega = 0$ are suppressed at large N in a similar manner to those resulting in hydrodynamic tails [30]. Alternatively, perhaps breaking T invariance allows decay channels that were not possible in [9].

The presence of the Goldstone boson reminds us that in our model, as in all such AdS/CFT systems, the U(1) 'electromagnetic' symmetry is not gauged in the boundary theory. Strictly, one might then consider our model to describe a charged superfluid rather than a superconductor. However, one can imagine 'weakly gauging' the global U(1) symmetry. That is to say, coupling the theory to photons with a weak coupling and hence ignoring all non-external photons interactions. This is analogous to what is done in the well-known BCS theory, where the electromagnetic interaction between electrons is screened and the effective interactions that drive superconductivity involve only phonons. Photons are also neglected in the standard treatment of p- and d-wave superconductors, see e.g. [33].

It is certainly of interest to look for possible embeddings of Yang-Mills theory in AdS into string or M theory, ideally consistent with the probe approximation. A natural way to get an SU(2) field in string theory is by using coincident D branes. In such a setup the probe limit will be admissible at weak string coupling (large N). For instance, the supersymmetric D3-D5 system, with N D3 branes and 2 D5 branes, admits a near horizon description as two probe D5 branes in $AdS_5 \times S^5$. The probe branes lie on an AdS_4 in AdS_5 and have an SU(2) gauge field on their worldvolume, thus precisely realising our setup. Although the remaining directions of the D5 branes form an S^2 in S^5 , they are stable because the mass of the slipping mode is above the Breitenlohner-Freedman bound.

The Lagrangian (2.3) can also be uplifted to eleven dimensional supergravity on $AdS_4 \times S^7$ using [31], following [32]. However, this lift gives $g^2 L^2 = \kappa_4^2/4$ (at least according to [31]).⁶ This appears to be below the critical value $g^2 L^2 \approx \kappa_4^2$ needed for a superconducting instability to occur [10]. Therefore, this lift does not realise the physics of interest.

4. Discussion and stability

It was shown in [25] that, at least near to the critical temperature, the isotropic superconducting phase is unstable to a perturbation breaking rotational invariance. This was

⁵We'd like to thank Dam Son for drawing this possibility to our attention.

⁶In fact [31] and [32] differ by a factor of 2 for $g^2 L^2$. Both values are below the critical value for superconductivity. We thank Gubser and Pufu for bringing these facts to our attention.

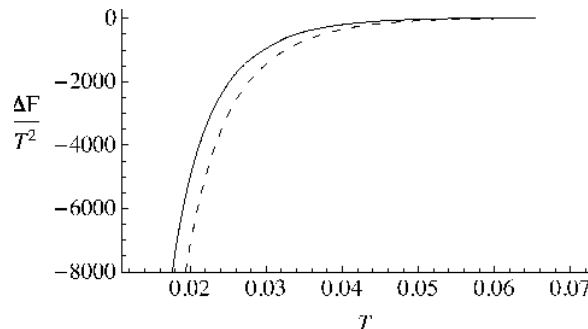


Figure 4: Comparison of free energies for temperatures below T_c . The anisotropic phase is a dashed line. It always has a lower free energy than the isotropic phase.

achieved by searching for dynamical instabilities, that is, normalisable modes that grow exponentially in time. Their postulated endpoint of this instability is an anisotropic phase with background ansatz

$$A = \phi(r) dt\tau^3 + w(r) dx\tau^1. \tag{4.1}$$

This background breaks time reversal symmetry, as well as being incurably anisotropic. The anisotropy means that it is not possible to invariantly separate the standard and Hall conductivities. Therefore one should look for different signals of the time reversal breaking.

To confirm this instability and follow it down to lower temperatures, we have computed the free energies of the isotropic and anisotropic phases. The two backgrounds come into existence at the same critical temperature T_c . We have been able to study the backgrounds down to $T/T_c \approx 0.23$. Working in the grand canonical ensemble, that is, fixed chemical potential, to compute the free energy it is sufficient to simply compare the Yang-Mills action (2.3) evaluated on the solutions, with the same chemical potential (and not necessarily the same charges).

The result is shown in figure 4. For each phase we have plotted the difference in free energy compared to the phase without a condensate. We see that the anisotropic phase appears to be favoured at all temperatures.

The mode about the isotropic black hole leading to an instability does not mix with the modes we have been considering. Therefore the electrical response of the theory remains well defined at the level of linear response. The phase we have studied is interesting therefore as a model of a strongly coupled isotropic superconductor with a pseudogap and spontaneous time reversal breaking. These are four properties (strongly coupled, isotropic, pseudogapped and T-non-invariant) that are shared by superconductors of experimental interest. It would of course be interesting to identify other experimental quantities that are amenable to computation in this model.

Further on the question of instability, one should note that there are other backgrounds than (2.7) and (4.1) that are possible, and which are natural candidates for the dominant state. After allowing for gauge transformations it seems that the general ansatz has three independent radial functions. This more general ansatz deserves study.

The main results of this paper have been to exhibit a pseudogap and a Hall conductivity in an isotropic phase of the holographic superconductor proposed in [10]. A primary question for future work is to explain the pseudogap in this model. We noted that two natural candidates are firstly non s -wave pairing and secondly intermediate states of massless Goldstone bosons.

A second question is to find upliftings of the Yang-Mills model into string and M theory. Probe D branes are a natural way to engineer nonabelian symmetries and will furthermore be consistent with the probe limit.

Thirdly, there has been recent work on the Hall conductivity of nonconventional superconductors, e.g. [33–37]. Some of these have been motivated by recent experiments on Sr_2RuO_4 , which is thought to have p -wave pairing. A qualitative difference between most of those results and ours is that they find a vanishing Hall conductivity when the spatial wavevector $k = 0$. We suspect that this difference is due to the fact that our condensate is not rotationally invariant without also rotating the internal $U(1)$. It would certainly be interesting to look for applications of our results to the physics of Sr_2RuO_4 or other materials.

Finally, we observed that the zero temperature gap was $2\Delta/T_c \approx 8$. This is very close to the value found in the holographic superconductor studied in [9]. It would be interesting to study more examples and see to what extent this value is universal for holographic superconductors.

Acknowledgments

We would like to thank Gary Horowitz for many helpful comments. It is also a pleasure to acknowledge discussions with Leon Balents, Matthew Fisher, David Mateos, Rafael Porto and Dam Son. We are grateful to Gubser and Pufu for their comments on the first preprint version of this paper. This research was supported in part by the National Science Foundation under Grants No. PHY05-51164 and PHY-0555669.

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